Theory of equations

Relation between roots and coefficients of a polynomial equations

Let α, β, γ be the roots of $x^3+px^2+qx+r=0$.
Then, writing the expression x^3+px^2+qx+r in the terms of α, β and γ gives $(x-\alpha)(x-\beta)(x-\gamma)$.

$$\therefore x^3+px^2+qx+r = (x-\alpha)(x-\beta)(x-\gamma)$$
$$= (x^2-[\alpha+\beta]x+\alpha\beta)(x-\gamma)$$
$$= x^3-(\alpha+\beta)x^2+\alpha\beta x-\gamma x^2 + (\alpha+\beta)\gamma x - \alpha\beta\gamma$$
$$= x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$$

From equating coefficients, we get

(a) $\alpha+\beta+\gamma = -p$ (b) $\alpha\beta+\beta\gamma+\gamma\alpha = q$ (c) $\alpha\beta\gamma = -r$.

This, of course, applies to a cubic equation. Let us extend this to a more general equation. In general, if $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of the equation

$$p_0x^n + p_1x^{n-1} + px^{n-2} + \dots + p_{n-1}x + p_n = 0$$

where $p_0 \neq 0$, then

(i) Sum of the roots $= \dfrac{-p_1}{p_0}$

(ii) Sum of products if the roots, three at a time $= \dfrac{-p_3}{p_0}$

(iii) Sum of products of the roots, n at a time

$$= (-1)^n \cdot \frac{p_n}{p_0}$$

The Factor theorem says that if a polynomial $p(x)$ has root r, then $x-r$ divides $p(x)$.

Since polynomial with complex coefficients always have exactly the same number of roots as its degree (counting multiplicity), and they have unique factorization, that means that if $p(x)$ is the polynomial

$x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_0$

and $r_1, r_2, \ldots, r_n$ are roots of $p(x)$, then we can factor $p(x)$ as

$$p(x) = x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_0$$
$$= (x-r_1)(x-r_2)\cdots(x-r_n).$$

When multiply out $(x-r_1)\cdots(x-r_n)$, and then put together the powers of x, we get a polynomial expression in which the coefficients are written in the terms of the roots. For example, to get the coefficient of x, note that we will get an x in the first binomial by each of the constants in the rest, giving us $(-1)^{n-1}(r_2 \cdots r_n)x$.

Another when we multiply the x in the second binomial by each of the constants in the rest; another when you multiply the x in the third binomial by the constants in the rest: etc

If we work this out, we will find that when we expand $(x-r_1)\cdots(x-r_n)$ and then group together the powers of x, you will have

* The coefficient of x^n is 1
* The coefficient of x^{n-1} is $-(r_1 + \cdots + r_n)$.
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* The coefficient of x is $(-1)^{n-1}(r_1 r_2 \cdots r_{n-1} + r_1 r_2 \cdots r_{n-2} r_n + \cdots + r_2 \cdots r_n)$
* The constant coefficient is $(-1)^n (r_1 \cdots r_n)$.

But for two polynomials to be equal they have to be equal coefficient by coefficient.

So that means that

$$p_1 = (-1)^1 (r_1 + \cdots + r_n)$$

$$p_2 = (-1)^2 (r_1 r_2 + r_1 r_3 + \cdots + r_1 r_n + r_2 r_3 + \cdots + r_{n-1} r_n)$$

$$p_3 = (-1)^3 (r_1 r_2 r_3 + r_1 r_2 r_4 + \cdots + r_1 r_2 r_n + r_1 r_3 r_4 + \cdots + r_{n-2} r_{n-1} r_n)$$

$$\vdots$$

$$p_n = (-1)^n (r_1 \cdots r_n).$$

i.e.

- The sum of the roots is $-p_1$;
- The sum of all products of two roots is $(-1)^2 p_2$;
- The sum of all products of three roots is $(-1)^3 p_3$;
- $\vdots$
- The sum of all products of $n-1$ roots is $(-1)^{n-1} p_{n-1}$;
- The product of all roots is $(-1)^n p_n$.

How about a polynomial that is not monic,

$$p(x) = p_0 x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_n, \quad p_0 \neq 0 \ ?$$

Note that r is a root of $p(x)$ if and only if it is a root of

$$P(x) = \frac{1}{p_0} p(x) = x^n + \frac{p_1}{p_0} x^{n-1} + \cdots + \frac{p_{n-1}}{p_0} x + \frac{p_n}{p_0}.$$

So the argument above applies of $P(x)$.